PLANE PROBLEM ON OSCILLATION OF A BODY UNDER TWO SURFACE-SEPARATING LIQUIDS

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The problem on steady oscillations of a body of arbitrary shape under a free surface of an infinitely deep liquid has been solved by Kochin [1]. The same problem for finite depth has been investigated by Haskind [2], using Kochin's method.

Here we investigate a plane problem of wave motions induced by oscillations of a body under a surface of separation of two liquids, by Kochin's method; the layer of the lighter upper liquid of finite thickness has a free surface, and the lower liquid has an inifinite depth.

1. Statement of the problem. Let the body oscillate periodically under the boundary of separation (Fig. 1). We will investigate infinitely small oscillations of a body, making usual assumptions of linear wave theory. We assume that waves propagate on both sides of the body, so that the liquid velocities are everywhere bounded and approach zero as $y' \to -\infty$. The boundary conditions on the free boundary, on the boundary of separation, and on the contour of body C, will be transferred on the lines $y' = 0$, $y' = -d$ and on the contour C respectively; this last is assumed to be stationary.

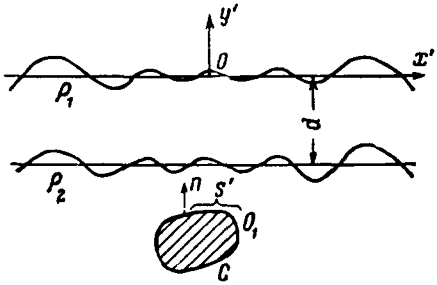

Fig. 1.

Assuming a potential motion, in this analysis we introduce the velocity potentials $\Phi_j'(x', y', t')$, stream functions $\Psi_j'(x', y', t')$ and the complex potentials

$$W_j'(z', t') = \Phi_j'(x', y', t') + i\Psi_j'(x', y', t') \qquad (1.1)$$

where $z' = x' + iy'$ and the index j is equal to 1 and 2 for the upper and lower liquid respectively.

By usual means we obtain the condition on the free boundary

$$\left[\frac{\partial^2\Phi_1'}{\partial t'^2} + g\frac{\partial\Phi_1'}{\partial y'}\right]_{y'=0} = 0 \qquad (1.2)$$

and two conditions on the boundary of separation

$$\left[\frac{\partial\Phi_1'}{\partial y'} - \frac{\partial\Phi_2'}{\partial y'}\right]_{y'=-d} = 0 \qquad (1.3)$$

$$\left[\left(\frac{\partial^2\Phi_1'}{\partial t'^2} + g\frac{\partial\Phi_1'}{\partial y'}\right) - \frac{\rho_2}{\rho_1}\left(\frac{\partial^2\Phi_2'}{\partial t'^2} + g\frac{\partial\Phi_2'}{\partial y'}\right)\right]_{y'=-d} = 0 \qquad (1.4)$$

The equations of the free boundary and of the boundary of separation have the form

$$\delta_1'(x') = -\frac{1}{g}\left[\frac{\partial\Phi_1'}{\partial t'}\right]_{y'=0}, \qquad \delta_2'(x') = \frac{1}{g(\rho_2-\rho_1)}\left[\rho_1\frac{\partial\Phi_1'}{\partial t'} - \rho_2\frac{\partial\Phi_2'}{\partial t'}\right]_{y'=-d} \qquad (1.5)$$

where $\delta_1'(x')$ is measured from the axis x', and $\delta_2'(x')$ from the line $y' = -d$.

Since boundary conditions are linear we may consider only purely harmonic oscillations of the body with frequency k, determined by the formula

$$v_n' = v_{n1}'(s')\cos kt' + v_{n2}'(s')\sin kt' = v_n'(s', t')$$

where v_n' is the normal velocity component of one of the points of the contour C; to that point there corresponds an arc length s' measured from some fixed point on C.

Then for function $\Phi_2'(x', y', t')$ we have the condition of streamline flow on C

$$\frac{\partial\Phi_2'}{\partial n} = v_n'(s', t') \qquad (1.6)$$

Assuming the oscillations of fluids to be steady, we take

$$\Phi_j'(x', y', t') = \varphi_{j1}'(x', y')\cos kt' + \varphi_{j2}'(x', y')\sin kt' \qquad (1.7)$$

where $\phi_{jm}'(x', y') = \operatorname{Re} w_{jm}'(z')$ if m is equal to 1 and 2. We will then have

$$W_j'(z',\ t') = w_{j1}'(z')\cos kt' + w_{j2}'(z')\sin kt' \tag{1.8}$$

The boundary conditions (1.2), (1.3), (1.4), (1.6) will be rewritten as follows:

$$\begin{gathered}
\left[g\frac{\partial\varphi_{1m}'}{\partial y'} - k^2\varphi_{1m}'\right]_{y'=0} = 0, \qquad \left[\frac{\partial\varphi_{1m}'}{\partial y'} - \frac{\partial\varphi_{2m}'}{\partial y'}\right]_{y'=-d} = 0 \\
\left[\left(g\frac{\partial\varphi_{1m}'}{\partial y'} - k^2\varphi_{1m}'\right) - \frac{\rho_2}{\rho_1}\left(g\frac{\partial\varphi_{2m}'}{\partial y'} - k^2\varphi_{2m}'\right)\right]_{y'=-d} = 0 \\
\frac{\partial\varphi_{2m}'}{\partial n} = v_{nm}'\ (s')
\end{gathered} \tag{1.9}$$

We introduce dimensionless quantities, denoting

$$\begin{gathered}
z' = zd, \qquad W_j' = W_j kd^2, \qquad w_{jm}' = w_{jm}' kd^2, \\
t' = \frac{t}{k}, \quad \frac{\rho_2}{\rho_1} = \rho_2^\circ, \qquad \frac{k^2 d}{g} = \nu
\end{gathered} \tag{1.10}$$

Denoting by E_1 the region occupied by the upper liquid, and by E_2 the region occupied by the lower liquid, we can formulate the problem in the following manner.

It is necessary to determine functions $w_{1m}^{\circ}(z)$ and $w_{2m}^{\circ}(z)$, which are analytical in regions E_1 and E_2 respectively, and which satisfy the conditions

$$1^\circ.\quad \frac{\partial\varphi_{1m}^\circ}{\partial y} - \nu\varphi_{1m}^\circ = 0 \qquad \text{for } y = 0 \tag{1.11}$$

$$2^\circ.\quad \frac{\partial\varphi_{1m}^\circ}{\partial y} - \frac{\partial\varphi_{2m}^\circ}{\partial y} = 0 \qquad \text{for } y = -1 \tag{1.12}$$

$$3^\circ.\quad \left(\frac{\partial\varphi_{1m}^\circ}{\partial y} - \nu\varphi_{1m}^\circ\right) - \rho_2^\circ\left(\frac{\partial\varphi_{2m}^\circ}{\partial y} - \nu\varphi_{2m}^\circ\right) = 0 \qquad \text{for } y = -1 \tag{1.13}$$

4. On the free boundary and on the boundary of separation the waves move out on both sides of the contour of body C.

5. In regions E_1 and E_2 outside the contour C the velocities are bounded and approach zero as $y \to -\infty$.

$$6^\circ.\quad \frac{\partial\varphi_{2m}^\circ}{\partial n} = v_{nm}(s) \qquad \text{on} \qquad C \tag{1.14}$$

Using relations (1.7), (1.11), (1.2) and (1.13), the equations of the free boundary and of the boundary of separation can be written in a final form

$$\delta_1(x) = -\operatorname{Im}\left[\frac{dw_{11}^\circ}{dz}\sin t - \frac{dw_{12}^\circ}{dz}\cos t\right]_{y=0} \tag{1.15}$$

$$\delta_2(x) = \operatorname{Im}\left[\frac{dw_{j1}^\circ}{dz}\sin t - \frac{dw_{j2}^\circ}{dz}\cos t\right]_{y=-1} \tag{1.16}$$

where j is equal to 1 or to 2.

2. The case of a pulsating vortex and a source. Assume at some point $\zeta = \xi + i\eta$ of the region $-1 > \operatorname{Im} z > -\infty$ a pulsating vortex of intensity $(\Gamma_1 \cos t + \Gamma_2 \sin t)$. Then from functions $w_{jm}^\circ(z)$ it is possible to separate the singularities at point ζ

$$w_{jm}^\circ(z) = w_{jm}(z) + F_m(z) \tag{2.1}$$

where $w_{1m}(z)$ and $w_{2m}(z)$ are functions, analytical in regions $0 > \operatorname{Im} z > -1$ and $-1 > \operatorname{Im} z > -\infty$ respectively, whereby $w_{jm} = \phi_{jm} + i\psi_{im}$ and

$$F_m(z) = \frac{\Gamma_m}{2\pi i}\ln\frac{z-\zeta}{z-\bar\zeta} \tag{2.2}$$

Differentiating equations (1.11) and (1.13) with respect to x, we can rewrite the first three boundary conditions from Section 1 in the form

$$\left[\frac{\partial^2\varphi_{1m}}{\partial x\partial y} - \nu\frac{\partial\varphi_{1m}}{\partial x}\right]_{y=0} = f_{1m}(x) \tag{2.3}$$

$$\left[\frac{\partial\varphi_{1m}}{\partial y} - \frac{\partial\varphi_{2m}}{\partial y}\right]_{y=-1} = 0 \tag{2.4}$$

$$\left[\left(\frac{\partial^2\varphi_{1m}}{\partial x\partial y} - \nu\frac{\partial\varphi_{1m}}{\partial x}\right) - \rho_2^\circ\left(\frac{\partial^2\varphi_{2m}}{\partial x\partial y} - \nu\frac{\partial\varphi_{2m}}{\partial x}\right)\right]_{y=-1} = (1-\rho_2^\circ) f_{2m}(x) \tag{2.5}$$

where

$$f_{1m}(x) = \operatorname{Im}\left[\frac{d^2F_m}{dz^2} + i\nu\frac{dF_m}{dz}\right]_{y=0}, \qquad f_{2m}(x) = \operatorname{Im}\left[\frac{d^2F_m}{dz^2} + i\nu\frac{dF_m}{dz}\right]_{y=-1} \tag{2.6}$$

Introducing the expressions for $F_m(z)$ in formulas (2.6) and using the known equations

$$\frac{|y|}{x^2+y^2} = \int_0^\infty e^{-\lambda|y|}\cos\lambda x\,d\lambda, \qquad \frac{x^2-y^2}{(x^2+y^2)^2} = -\int_0^\infty e^{-\lambda|y|}\lambda\cos\lambda x\,d\lambda$$

we obtain

$$f_{1m}(x) = -\frac{\Gamma_m}{\pi}\nu\int_0^\infty e^{\lambda\eta}\cos\lambda(x-\xi)\,d\lambda$$
$$f_{2m}(x) = \frac{\Gamma_m}{\pi}\int_0^\infty e^{\lambda\eta}(\lambda\,\sinh\lambda+\nu\cosh\lambda)\cos\lambda(x-\xi)\,d\lambda \tag{2.7}$$

We seek the solution in the form of Fourier integrals

$$w_{1m}(z) = \Gamma_m\int_0^\infty \{[A(\lambda)+iB(\lambda)]e^{-i\lambda(z-\overline{\zeta})}+[C(\lambda)+iD(\lambda)]e^{i\lambda(z-\zeta)}\}\frac{d\lambda}{\lambda} \tag{2.8}$$
$$w_{2m}(z) = \Gamma_m\int_0^\infty [E(\lambda)+iG(\lambda)]e^{-i\lambda(z-\overline{\zeta})}\frac{d\lambda}{\lambda}$$

using condition 5 to ensure boundedness of functions $dw_{jm}^{\,0}/dz$.

Introducing the functions w_{jm} into conditions (2.3), (2.4), (2.5) from (2.8) and using relationships (2.7), we obtain the equations for determination of coefficients, from which we will find

$$A(\lambda)=0,\qquad B(\lambda)=\frac{1}{\pi\nu}\left\{\frac{L(\lambda)[2\nu-\varkappa(\lambda-\nu)]}{(\lambda-\nu)T(\lambda)}-\frac{\varkappa}{4}[(\lambda+\nu)e^{2\lambda}-(\lambda-\nu)]\right\}=B$$

$$C(\lambda)=0,\qquad D(\lambda)=\frac{\varkappa}{\pi\nu}\left\{\frac{L(\lambda)}{T(\lambda)}e^{-2\lambda}+\frac{1}{4}[(\lambda+\nu)-(\lambda-\nu)e^{-2\lambda}]\right\}=D \tag{2.9}$$

$$E(\lambda)=0,\qquad G(\lambda)=\frac{2L(\lambda)}{\pi(\lambda-\nu)T(\lambda)}=G$$

where

$$L(\lambda)=-\nu^2+\varkappa(\lambda^2\sinh^2\lambda-\nu^2\cosh^2\lambda)$$
$$T(\lambda)=2\nu+\varkappa[(\lambda+\nu)e^{-2\lambda}-(\lambda-\nu)]\qquad(\varkappa=\rho_2^{\circ}-1) \tag{2.10}$$

It is easy to see that for any $\nu>0$ the equation $T(\lambda)=0$ has one positive root $\lambda=\lambda_0$ and one only. Since the integrands for functions dw_{jm}/dz have two simple poles $\lambda=\nu$ and $\lambda=\lambda_0$, on the real positive semi-axis λ, we will take the Cauchy principal values of the integral.

2. To find the general solution, to the obtained functions $w_{jm}(z)$ we will add the potentials of the free waves

$$w_{1m}^{\infty}(z) = A^{\circ}_m e^{-i\nu(z-\bar{\zeta})} + B_m^{\circ} e^{-i\lambda_0(z-\bar{\zeta})} + C_m^{\circ} e^{i\lambda_0(z-\zeta)}$$
$$w_{2m}^{\infty}(z) = E_m^{\circ} e^{-i\nu(z-\bar{\zeta})} + G_m^{\circ} e^{-i\lambda_0(z-\bar{\zeta})} \tag{2.11}$$

where A_m°, B_m°, ..., G_m° are constants. Therefore

$$w_{1m}^{\circ}(z) = \frac{\Gamma_m}{2\pi i} \ln \frac{z-\zeta}{z-\bar{\zeta}} + i\Gamma_m \int_0^{\infty} [Be^{-i\lambda(z-\bar{\zeta})} +$$
$$+ De^{i\lambda(z-\zeta)}] \frac{d\lambda}{\lambda} + A_m^{\circ} e^{-i\nu(z-\bar{\zeta})} + B_m^{\circ} e^{-i\lambda_0(z-\bar{\zeta})} + C_m^{\circ} e^{i\lambda_0(z-\zeta)} \tag{2.12}$$
$$w_{2m}^{\circ}(z) = \frac{\Gamma_m}{2\pi i} \ln \frac{z-\zeta}{z-\bar{\zeta}} + i\Gamma_m \int_0^{\infty} Ge^{-i\lambda(z-\bar{\zeta})} \frac{d\lambda}{\lambda} + E_m^{\circ} e^{-i\nu(z-\bar{\zeta})} + G_m^{\circ} e^{-i\lambda_0(z-\bar{\zeta})}$$

The unknown constants will be determined from condition 4^0, according to which the waves propagate in both directions from the vortex.

To determine the asymptotic values of functions dw_{2m}/dz for $x < 0$, we will write them in the form

$$\frac{dw_{2m}}{dz} = \Gamma_m \int_{L_+} Ge^{-i\lambda(z-\bar{\zeta})}\, d\lambda + 2i\Gamma_m \frac{L(\nu)}{T(\nu)} e^{-i\nu(z-\bar{\zeta})} +$$
$$+ 2i\Gamma_m \frac{L(\lambda_0)}{(\lambda_0-\nu)\, T'(\lambda_0)} e^{-i\lambda_0(z-\bar{\zeta})}, \qquad x<0$$

where L_+ is the contour in the plane of the complex variable λ (Fig. 2), and

$$T'(\lambda_0) = \left(\frac{dT}{d\lambda}\right)_{\lambda=\lambda_0}$$

Integration by parts shows that the integral along the contour,

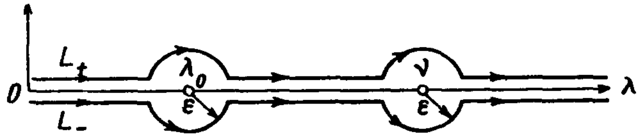

Fig. 2.

L_+ approaches 0 as $x \to -\infty$, consequently

$$\lim_{x\to-\infty} \left\{\frac{dw_{2m}}{dz} - [i\Gamma_m E_0 e^{-i\nu(z-\bar{\zeta})} + i\Gamma_m G_0 e^{-i\lambda_0(z-\bar{\zeta})}]\right\} = 0 \tag{2.13}$$

where

$$E_0 = \frac{2L(\nu)}{T(\nu)}, \qquad G_0 = \frac{2L(\lambda_0)}{(\lambda_0 - \nu)\,T'(\lambda_0)} \tag{2.14}$$

Utilizing the integral along L_-, we similarly obtain

$$\lim_{x\to+\infty}\left\{\frac{dw_{2m}}{dz} + [i\Gamma_m E_0 e^{-i\nu(z-\bar{\zeta})} + i\Gamma_m G_0 e^{-i\lambda_0(z-\bar{\zeta})}]\right\} = 0 \tag{2.15}$$

Inasmuch as the waves propagate in both directions from the vortex, the asymptotic expressions for the full complex velocity can be written in the form

$$\begin{aligned}&\lim_{x\to-\infty}\left\{\frac{dW_2}{dz} - [E_-^\circ e^{-i\nu(z-\bar{\zeta})-it} + G_-^\circ e^{-i\lambda_0(z-\bar{\zeta})-it}]\right\} = 0\\ &\lim_{x\to+\infty}\left\{\frac{dW_2}{dz} - [E_+^\circ e^{-i\nu(z-\bar{\zeta})+it} + G_+^\circ e^{-i\lambda_0(z-\bar{\zeta})+it}]\right\} = 0\end{aligned} \tag{2.16}$$

Taking formula (1.8) into account, we will have

$$\frac{dW_j}{dz} = \frac{dw_{j1}^\circ}{dz}\cos t + \frac{dw_{j2}^\circ}{dz}\sin t \tag{2.17}$$

Introducing the expressions (2.13) and (2.15) in the formula (2.17) and then comparing it with (2.16), we finally obtain

$$\begin{aligned}&E_1^\circ = -i\Gamma_2\frac{E_0}{\nu}, && E_2^\circ = i\Gamma_1\frac{E_0}{\nu}\\ &E_-^\circ = (i\Gamma_1 - \Gamma_2)E_0, && E_+^\circ = -(i\Gamma_1 + \Gamma_2)E_0\\ &G_1^\circ = -i\Gamma_2\frac{G_0}{\lambda_0}, && G_2^\circ = i\Gamma_1\frac{G_0}{\lambda_0}\\ &G_-^\circ = (i\Gamma_1 - \Gamma_2)G_0, && G_+^\circ = -(i\Gamma_1 + \Gamma_2)G_0\end{aligned} \tag{2.18}$$

Similarly it is possible to find the asymptotic expressions for dW/dz

$$\begin{aligned}&\lim_{x\to-\infty}\left\{\frac{dW_1}{dz} - [A_-^\circ e^{-i\nu(z-\bar{\zeta})-it} + B_-^\circ e^{-i\lambda_0(z-\bar{\zeta})-it} + C_-^\circ e^{i\lambda_0(z-\zeta)+it}]\right\} = 0\\ &\lim_{x\to+\infty}\left\{\frac{dW_1}{dz} - [A_+^\circ e^{-i\nu(z-\bar{\zeta})+it} + B_+^\circ e^{-i\lambda_0(z-\bar{\zeta})+it} + C_+^\circ e^{i\lambda_0(z-\zeta)-it}]\right\} = 0\end{aligned} \tag{2.19}$$

and also the values of the unknown constants

(2.20)

$$A_1^\circ = -i\Gamma_2\frac{A_0}{\nu}, \quad A_2^\circ = i\Gamma_1\frac{A_0}{\nu}, \quad A_-^\circ = (i\Gamma_1 - \Gamma_2)A_0, \quad A_+^\circ = -(i\Gamma_1 + \Gamma_2)A_0$$

$$B_1^\circ = -i\Gamma_2\frac{B_0}{\lambda_0}, \quad B_2^\circ = i\Gamma_1\frac{B_0}{\lambda_0}, \quad B_-^\circ = (i\Gamma_1 - \Gamma_2)B_0, \quad B_+^\circ = -(i\Gamma_1 + \Gamma_2)B_0$$

$$C_1^\circ = -i\Gamma_2\frac{C_0}{\lambda_0}, \quad C_2^\circ = i\Gamma_1\frac{C_0}{\lambda_0}, \quad C_-^\circ = (i\Gamma_1 + \Gamma_2)C_0, \quad C_+^\circ = -(i\Gamma_1 - \Gamma_2)C_0$$

where

$$A_0 = \frac{2L(\nu)}{T(\nu)} = E_0, \quad B_0 = \frac{L(\lambda_0)[2\nu - \varkappa(\lambda_0 - \nu)]}{\nu(\lambda_0 - \nu) T'(\lambda_0)}, \quad C_0 = \frac{\varkappa L(\lambda_0)}{\nu T'(\lambda_0)} e^{-2\lambda_0} \tag{2.21}$$

3. For a source of intensity ($Q_1 \cos t + Q_2 \sin t$), situated at the point $\zeta = \xi + i\eta$, it is possible by the same method to obtain complex potentials

$$w_{1m}^{\circ}(z) = \frac{Q_m}{2\pi} \ln(z-\zeta)(z-\bar{\zeta}) + Q_m \int_0^{\infty} [Be^{-i\lambda(z-\bar{\zeta})} - De^{i\lambda(z-\zeta)}] \frac{d\lambda}{\lambda} +$$

$$+ A_m^{\infty} e^{-i\nu(z-\bar{\zeta})} + B_m^{\infty} e^{-i\lambda_0(z-\bar{\zeta})} + C_m^{\infty} e^{i\lambda_0(z-\zeta)}, \quad w_{2m}^{\circ}(z) = \frac{Q_m}{2\pi} \ln(z-\zeta)(z-\bar{\zeta}) +$$

$$+ Q_m \int_0^{\infty} Ge^{-i\lambda(z-\bar{\zeta})} \frac{d\lambda}{\lambda} + E_m^{\infty} e^{-i\nu(z-\bar{\zeta})} + G_m^{\infty} e^{-i\lambda_0(z-\bar{\zeta})} \tag{2.22}$$

and the asymptotic expressions of complete complex velocities

$$\lim_{x\to-\infty} \left\{\frac{dW_1}{dz} - \left[A_-^{\infty} e^{-i\nu(z-\bar{\zeta})-it} + B_-^{\infty} e^{-i\lambda_0(z-\bar{\zeta})-it} + C_-^{\infty} e^{i\lambda_0(z-\zeta)+it}\right]\right\} = 0$$

$$\lim_{x\to+\infty} \left\{\frac{dW_1}{dz} - [A_+^{\infty} e^{-i\nu(z-\bar{\zeta})+it} + B_+^{\circ} e^{-i\lambda_0(z-\bar{\zeta})+it} + C_+^{\circ} e^{i\lambda_0(z-\zeta)-it}]\right\} = 0 \tag{2.23}$$

$$\lim_{x\to-\infty} \left\{\frac{dW_2}{dz} - [E_-^{\infty} e^{-i\nu(z-\bar{\zeta})-it} + G_-^{\infty} e^{-i\lambda_0(z-\bar{\zeta})-it}]\right\} = 0$$

$$\lim_{x\to+\infty} \left\{\frac{dW_2}{dz} - [E_+^{\infty} e^{-i\nu(z-\bar{\zeta})+it} + G_+^{\infty} e^{-i\lambda_0(z-\bar{\zeta})+it}]\right\} = 0 \tag{2.24}$$

where

$$\begin{aligned}
&A_1^{\infty} = -Q_2 \frac{A_0}{\nu}, \quad A_2^{\infty} = Q_1 \frac{A_0}{\nu}, \quad A_-^{\infty} = (Q_1 + iQ_2) A_0 \\
&B_1^{\infty} = -Q_2 \frac{B_0}{\lambda_0}, \quad B_2^{\infty} = Q_1 \frac{B_0}{\lambda_0}, \quad B_-^{\infty} = (Q_1 + iQ_2) B_0 \\
&C_1^{\infty} = Q_2 \frac{C_0}{\lambda_0}, \quad C_2^{\infty} = -Q_1 \frac{C_0}{\lambda_0}, \quad C_-^{\infty} = -(Q_1 - iQ_2) C_0 \\
&E_1^{\infty} = -Q_2 \frac{E_0}{\nu}, \quad E_2^{\infty} = Q_1 \frac{E_0}{\nu}, \quad E_-^{\infty} = (Q_1 + iQ_2) E_0 \\
&G_1^{\infty} = -Q_2 \frac{G_0}{\lambda_0}, \quad G_2^{\infty} = Q_1 \frac{G_0}{\lambda_0}, \quad G_-^{\infty} = (Q_1 + iQ_2) G_0
\end{aligned} \tag{2.25}$$

$$\begin{aligned}
&A_+^{\infty} = -(Q_1 - iQ_2) A_0, \quad E_+^{\infty} = -(Q_1 - iQ_2) E_0 \\
&B_+^{\infty} = -(Q_1 - iQ_2) B_0, \quad G_+^{\infty} = -(Q_1 - iQ_2) G_0 \\
&C_+^{\infty} = (Q_1 + iQ_2) C_0
\end{aligned}$$

3. On waves produced by oscillations of the body. 1. We will derive the basic formulas for the problem postulated in Section 1, assuming that the solution of this problem is already known; we will deal with the actual solution in Section 4.

We take point z in the region E_2 and draw two contours C_1 and C_∞ in such a manner that they are located entirely in region E_2, and that C_1 encloses the contour of body C but does not contain point z, whereas C_∞ encloses both the contour C_1 and the point z (Fig. 3). For functions single-valued in region E_2 $dw_{2m}{}^0/dz = v_{2m}(z)$, the Cauchy formulas are applicable

$$\bar{v}_{2m}(z) = \frac{1}{2\pi i}\int\limits_{C_1} \frac{\bar{v}_{2m}(\zeta)}{z-\zeta}\,d\zeta - \frac{1}{2\pi i}\int\limits_{C_\infty} \frac{\bar{v}_{2m}(\zeta)}{z-\zeta}\,d\zeta \tag{3.1}$$

where both contours C_1 and C_∞ are traced in the positive direction, and the bar over the letter indicates that a complex conjugate expression is taken. We designate

$$V_m(z) = \frac{1}{2\pi i}\int\limits_{C_\infty} \frac{\bar{v}_{2m}(\zeta)}{z-\zeta}\,d\zeta$$
$$U_m(z) = -\frac{1}{2\pi i}\int\limits_{C_\infty} \frac{\bar{v}_{2m}(\zeta)}{z-\zeta}\,d\zeta \tag{3.2}$$

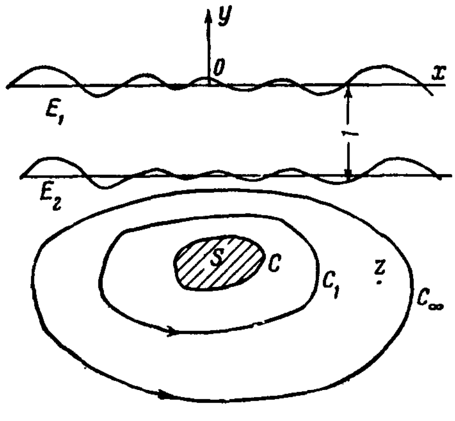

Fig. 3.

It is evident that $V_m(z)$ are analytical functions over the entire surface of the complex variable z outside the contour C_1, which can be drawn as close as desired to the contour of body C, and behave as $1/z$ at infinity. The functions $U_m(z)$ are analytical within the contour C_∞, which can be taken as close as desired to the line $y = -1$.

Therefore, it is possible to consider that the motion of the liquids is caused by vortex-sources of densities $v_{2m}(\zeta)$ distributed on the contour C_1.

We will utilize this fact to determine functions $v_{1m}(z) = dw^0_{1m}/dz$ and also to represent functions $U_m(z)$ in a different form.

Using formulas (2.12) and (2.22), we obtain complex velocities for the vortex-source of intensity $N_m = \Gamma_m + iQ_m$:

$$U_{1m}(z) = \frac{N_m}{2\pi i}\frac{1}{z-\zeta} - \frac{\overline{N}_m}{2\pi i}\frac{1}{z-\overline{\zeta}} + \overline{N}_m\int_0^\infty Be^{-i\lambda(z-\overline{\zeta})}\,d\lambda - N_m\int_0^\infty De^{i\lambda(z-\zeta)}d\lambda -$$
$$- i\,[\nu A_m^* e^{-i\nu(z-\overline{\zeta})} + \lambda_0 B_m^* e^{-i\lambda_0(z-\overline{\zeta})}] + i\lambda_0 C_m^* e^{i\lambda_0(z-\zeta)} \tag{3.3}$$

$$U_{2m}(z) = \frac{N_m}{2\pi i}\frac{1}{z-\zeta} - \frac{\overline{N}_m}{2\pi i}\frac{1}{z-\overline{\zeta}} + \overline{N}_m\int_0^\infty Ge^{-i\lambda(z-\overline{\zeta})}\,d\lambda - i\,[\nu E_m^* e^{-i\nu(z-\overline{\zeta})} +$$
$$+ \lambda_0 G_m^* e^{-i\lambda_0(z-\overline{\zeta})}] \tag{3.4}$$

where

$$A_1^* = -i\overline{N}_2\frac{A_0}{\nu}, \quad A_2^* = i\overline{N}_1\frac{A_0}{\nu}, \quad B_1^* = -i\overline{N}_2\frac{B_0}{\lambda_0}, \quad B_2^* = i\overline{N}_1\frac{B_0}{\lambda_0}$$
$$C_1^* = -iN_2\frac{C_0}{\lambda_0}, \quad C_2^* = iN_1\frac{C_0}{\lambda_0}, \quad E_1^* = -i\overline{N}_2\frac{E_0}{\nu}$$
$$E_2^* = i\overline{N}_1\frac{E_0}{\nu}, \quad G_1^* = -i\overline{N}_2\frac{G_0}{\lambda_0}, \quad G_2^* = i\overline{N}_1\frac{G_0}{\lambda_0} \tag{3.5}$$

Considering the equality

$$\frac{1}{z-\overline{\zeta}} = i\int_0^\infty e^{-i\lambda(z-\overline{\zeta})}\,d\lambda$$

true for $y + \eta > 0$, taking in formulas (3.3) and (3.4) $N_m = v_{2m}(\zeta)\,d\zeta$ and integrating along the contour C_1, we obtain the complex velocities in regions E_1 and E_2

$$\overline{v}_{11}(z) = V_1(z) + \int_{C_1} v_{21}(\zeta)\Big[-\frac{1}{2\pi}\int_0^\infty e^{-i\lambda(z-\overline{\zeta})}\,d\lambda + \int_0^\infty Be^{-i\lambda(z-\overline{\zeta})}\,d\lambda\Big]d\overline{\zeta} -$$
$$-\int_{C_1}\overline{v}_{21}(\zeta)\Big[\int_0^\infty De^{i\lambda(z-\zeta)}\,d\lambda\Big]d\zeta - \int_{C_1} v_{22}(\zeta)\,[A_0e^{-i\nu(z-\overline{\zeta})} + B_0e^{-i\lambda_0(z-\overline{\zeta})}]\,d\overline{\zeta} +$$
$$+ C_0\int_{C_1}\overline{v}_{22}(\zeta)\,e^{i\lambda_0(z-\zeta)}\,d\zeta \tag{3.6}$$

$$\overline{v}_{12}(z) = V_2(z) + \int_{C_1} v_{22}(\zeta)\Big[-\frac{1}{2\pi}\int_0^\infty e^{-i\lambda(z-\overline{\zeta})}\,d\lambda + \int_0^\infty Be^{-i\lambda(z-\overline{\zeta})}\,d\lambda\Big]d\overline{\zeta} -$$
$$-\int_{C_1}\overline{v}_{22}(\zeta)\Big[\int_0^\infty De^{i\lambda(z-\zeta)}\,d\lambda\Big]d\zeta + \int_{C_1} v_{21}(\zeta)\,[A_0e^{-i\nu(z-\overline{\zeta})} + B_0e^{-i\lambda_0(z-\overline{\zeta})}]\,d\overline{\zeta} -$$
$$- C_0\int_{C_1}\overline{v}_{21}(\zeta)\,e^{i\lambda_0(z-\zeta)}\,d\zeta$$

$$\bar{v}_{21}(z)=V_1(z)+\int_{C_1} v_{21}(\zeta)\Big[-\frac{1}{2\pi}\int_0^\infty e^{-i\lambda(z-\bar{\zeta})}\,d\lambda+\int_0^\infty Ge^{-i\lambda(z-\bar{\zeta})}\,d\lambda\Big]d\bar{\zeta}-$$

$$-\int_{C_1} v_{22}(\zeta)\,[E_0e^{-i\nu(z-\bar{\zeta})}+G_0e^{-i\lambda_0(z-\bar{\zeta})}]\,d\bar{\zeta}$$

$$\bar{v}_{22}(z)=V_2(z)+\int_{C_1} v_{22}(\zeta)\Big[-\frac{1}{2\pi}\int_0^\infty e^{-i\lambda(z-\bar{\zeta})}\,d\lambda+$$

$$+\int_0^\infty Ge^{-i\lambda(z-\bar{\zeta})}\,d\lambda\Big]d\bar{\zeta}+\int_{C_1} v_{21}(\zeta)\,[E_0e^{-i\nu(z-\bar{\zeta})}+G_0e^{-i\lambda_0(z-\bar{\zeta})}]\,d\bar{\zeta} \qquad (3.7)$$

We introduce the complex conjugate functions. at real λ,

$$H_m(\lambda)=\int_{C_1}\bar{v}_{2m}(\zeta)\,e^{-i\lambda\zeta}\,d\zeta,\quad \overline{H}_m(\lambda)=\int_{C_1} v_{2m}(\zeta)\,e^{i\lambda\bar{\zeta}}d\bar{\zeta} \qquad (3.8)$$

Interchanging the order of integration in (3.6) and (3.7), we obtain

$$\bar{v}_{11}(z)=V_1(z)+\int_0^\infty\Big[\overline{H}_1(\lambda)\,e^{-i\lambda z}\Big(-\frac{1}{2\pi}+B\Big)-H_1(\lambda)\,e^{i\lambda z}D\Big]d\lambda-A_0\overline{H}_2(\nu)\,e^{-i\nu z}-,$$

$$-B_0\overline{H}_2(\lambda_0)\,e^{-i\lambda_0 z}+C_0H_2(\lambda_0)\,e^{i\lambda_0 z} \qquad (3.9)$$

$$\bar{v}_{12}(z)=V_2(z)+\int_0^\infty\Big[\overline{H}_2(\lambda)\,e^{-i\lambda z}\Big(-\frac{1}{2\pi}+B\Big)-H_2(\lambda)\,e^{i\lambda z}\,D\Big]d\lambda+$$

$$A_0\overline{H}_1(\nu)\;+\;e^{-i\nu z}+B_0\overline{H}_1(\lambda_0)\,e^{-i\lambda_0 z}-C_0H_1(\lambda_0)\,e^{i\lambda_0 z}$$

$$\bar{v}_{21}\;(z)\;=\;V_1\;+\;\int_0^\infty\overline{H}_1(\lambda)\,e^{-i\lambda z}\Big(G-\frac{1}{2\pi}\Big)d\lambda-E_0\overline{H}_2(\nu)\,e^{-i\nu z}-G_0\overline{H}_2(\lambda_0)\,e^{-i\lambda_0 z}$$

$$\bar{v}_{22}\;(z)\;=\;V_2\;+\;\int_0^\infty\overline{H}_2(\lambda)\,e^{-i\lambda z}\Big(G-\frac{1}{2\pi}\Big)d\lambda+E_0\overline{H}_1(\nu)\,e^{-i\nu z}+G_0\overline{H}_1(\lambda_0)\,e^{-i\lambda_0 z}$$

where

(3.10)

$$V_1=V_1(z)=\int_0^\infty\overline{H}_1(\lambda)\,e^{-i\lambda z}\Big(-\frac{1}{2\pi}+G\Big)d\lambda-E_0\overline{H}_2(\nu)\,e^{-i\nu z}-G_0\overline{H}_2(\lambda_0)\,e^{-i\lambda_0 z}$$

$$V_2=V_2(z)=\int_0^\infty\overline{H}_2(\lambda)^{-i\lambda z}\Big(-\frac{1}{2\pi}+G\Big)d\lambda+E_0\overline{H}_1(\nu)^{-i\nu z}+G_0\overline{H}_1(\lambda_0)\,e^{-i\lambda_0 z} \qquad (3.11)$$

2. We will determine the waves on the free boundary and on the boundary of separation at a distance from the body. On the basis of formulas (2.16), (2.19), (2.23) and (2.24) we can write the asymptotic expressions of the complete complex velocities for a vortex-source of intensity $N_1 \cos t + N_2 \sin t$ (where $N_m = \Gamma_m + iQ_m$):

$$\lim_{x\to-\infty} \{u_1(z) - [A_- e^{-i\nu(z-\bar\zeta)-it} + B_- e^{-i\lambda_0(z-\bar\zeta)-it} + C_- e^{i\lambda_0(z-\zeta)+it}]\} = 0$$

$$\lim_{x\to+\infty} \{u_1(z) - [A_+ e^{-i\nu(z-\bar\zeta)+it} + B_+ e^{-i\lambda_0(z-\bar\zeta)+it} + C_+ e^{i\lambda_0(z-\zeta)-it}]\} = 0 \qquad (3.12)$$

$$\lim_{x\to-\infty} \{u_2(z) - [E_- e^{-i\nu(z-\bar\zeta)-it} + G_- e^{-i\lambda_0(z-\bar\zeta)-it}]\} = 0 \qquad (3.13)$$

$$\lim_{x\to+\infty} \{u_2(z) - [E_+ e^{-i\nu(z-\bar\zeta)+it} + G_+ e^{-i\lambda_0(z-\bar\zeta)+it}]\} = 0$$

Here

$$A_- = (i\bar N_1 - \bar N_2) A_0, \quad A_+ = -(i\bar N_1 + \bar N_2) A_0, \quad B_+ = -(i\bar N_1 + \bar N_2) B_0$$

$$B_- = (i\bar N_1 - \bar N_2) B_0$$

$$C_- = (iN_1 + N_2) C_0, \quad C_+ = -(iN_1 - N_2) C_0, \quad E_+ = -(i\bar N_1 + \bar N_2) E_0$$

$$E_- = (i\bar N_1 - \bar N_2) E_0$$

$$G_- = (i\bar N_1 - \bar N_2) G_0, \quad G_+ = -(i\bar N_1 + \bar N_2) G_0$$

Taking $N_m = v_{2m}(\zeta)\, d\zeta$, integrating the expressions (3.12) and (3.13) along the contour C_1, and using formulas (3.8), we obtain

$$\lim_{x\to-\infty} \Big\{\frac{dW_1}{dz} - iA_0[\bar H_1(\nu) + i\bar H_2(\nu)]\, e^{-i(\nu z+t)} - iB_0[\bar H_1(\lambda_0) + i\bar H_2(\lambda_0)] e^{-i(\lambda_0 z+t)} -$$

$$- iC_0[H_1(\lambda_0) - iH_2(\lambda_0)]\, e^{i(\lambda_0 z+t)}\Big\} = 0 \qquad (3.14)$$

$$\lim_{x\to+\infty} \Big\{\frac{dW_1}{dz} + iA_0[\bar H_1(\nu) - i\bar H_2(\nu)]\, e^{-i(\nu z-t)} + iB_0[\bar H_1(\lambda_0) - i\bar H_2(\lambda_0)]\, e^{-i(\lambda_0 z-t)} +$$

$$+ iC_0[H_1(\lambda_0) + iH_2(\lambda_0)] e^{i(\lambda_0 z-t)}\Big\} = 0$$

$$\lim_{x\to-\infty} \Big\{\frac{dW_2}{dz} - iE_0[\bar H_1(\nu) + i\bar H_2(\nu)]\, e^{-i(\nu z+t)} -$$

$$- iG_0[\bar H_1(\lambda_0) + i\bar H_2(\lambda_0)]\, e^{-i(\lambda_0 z+t)}\Big\} = 0 \qquad (3.15)$$

$$\lim_{x\to+\infty} \Big\{\frac{dW_2}{dz} + iE_0[\bar H_1(\nu) - i\bar H_2(\nu)]\, e^{-i(\nu z-t)} +$$

$$+ iG_0[\bar H_1(\lambda_0) - i\bar H_2(\lambda_0)] e^{-i(\lambda_0 z-t)}\Big\} = 0$$

Taking formula (2.17) into account, it is possible from (3.14) and (3.15) to find the asymptotic expressions for functions dw_{jm}^0/dz, the introduction of which in formulas (1.15) and (1.16) makes it possible to determine the wave profiles on the free boundary and on the boundary of separation at a distance from the body:

$$\begin{aligned}
\delta_1(x) &\approx \operatorname{Im}\{A_0[\overline{H}_1(\nu)+i\overline{H}_2(\nu)]\,e^{-i(\nu x+t)}+\\
&\quad+(B_0+C_0)[\overline{H}_1(\lambda_0)+i\overline{H}_2(\lambda_0)]\,e^{-i(\lambda_0 x+t)}\}, \quad x\to-\infty\\
\delta_1(x) &\approx \operatorname{Im}\{A_0[\overline{H}_1(\nu)-i\overline{H}_2(\nu)]\,e^{-i(\nu x-t)}+\\
&\quad+(B_0+C_0)[\overline{H}_1(\lambda_0)-i\overline{H}_2(\lambda_0)]\,e^{-i(\lambda_0 x-t)}\}, \quad x\to+\infty
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
\delta_2(x) &\approx \operatorname{Im}\{E_0e^{-\nu}[\overline{H}_1(\nu)+i\overline{H}_2(\nu)]\,e^{-i(\nu x+t)}+\\
&\quad+G_0e^{-\lambda_0}[H_1(\lambda_0)+i\bar{H}_2(\lambda_0)]\,e^{-i(\lambda_0 x+t)}\}, \quad x\to-\infty\\
\delta_2(x) &\approx \operatorname{Im}\{E_0e^{-\nu}[\overline{H}_1(\nu)-i\overline{H}_2(\nu)]\,e^{-i(\nu x-t)}+\\
&\quad+G_0e^{-\lambda_0}[\bar{H}_1(\lambda_0)-i\overline{H}_2(\lambda_0)]\,e^{-i(\lambda_0 x-t)}\}, \quad x\to+\infty
\end{aligned} \tag{3.17}$$

Denoting

$$\begin{aligned}
\alpha_{1-} &= |A_0|\,|\overline{H}_1(\nu)+i\overline{H}_2(\nu)|, & \beta_{1-} &= |B_0+C_0|\,|\overline{H}_1(\lambda_0)+i\overline{H}_2(\lambda_0)|\\
\alpha_{1+} &= |A_0|\,|\overline{H}_1(\nu)-i\overline{H}_2(\nu)|, & \beta_{1+} &= |B_0+C_0|\,|\overline{H}_1(\lambda_0)-i\overline{H}_2(\lambda)|\\
\alpha_{2-} &= e^{-\nu}\,|E_0|\,|\overline{H}_1(\nu)+i\overline{H}_2(\nu)|, & \beta_{2-} &= e^{-\lambda_0}\,|G_0|\,|\overline{H}_1(\lambda_0)+i\overline{H}_2(\lambda_0)|\\
\alpha_{2+} &= e^{-\nu}\,|E_0|\,|\overline{H}_1(\nu)-i\overline{H}_2(\nu)|, & \beta_{2+} &= e^{-\lambda_0}\,|G_0|\,|\overline{H}_1(\lambda_0)-i\overline{H}_2(\lambda_0)|
\end{aligned} \tag{3.18}$$

it is possible to state the following. On the free boundary and on the boundary of separation waves are propagated on both sides of the body, and their profiles are due to superposition of two wave shapes, waves with lengths $2\pi/\nu$ and $2\pi/\lambda_0$. The amplitudes of waves travelling in the direction of decreasing and increasing x are respectively equal to α_{j-} and α_{j+} for waves of length $2\pi/\nu$ and are equal to β_{j-} and β_{j+} for waves of length $2\pi/\lambda_0$, where $j = 1$ for the free boundary and $j = 2$ for the boundary of separation.

Knowing the value

$$B_0+C_0=-\frac{2\varkappa L(\lambda_0)}{(\lambda_0-\nu)\,T'(\lambda_0)}\,e^{-2\lambda_0}$$

we can write the ratio of amplitudes:

$$\frac{\alpha_{2-}}{\alpha_{1-}}=e^{-\nu}, \qquad \frac{\alpha_{2+}}{\alpha_{1+}}=e^{-\nu}, \qquad \frac{\beta_{2-}}{\beta_{1-}}=\frac{e^{\lambda_0}}{\varkappa}, \qquad \frac{\beta_{2+}}{\beta_{1+}}=\frac{e^{\lambda_0}}{\varkappa} \tag{3.19}$$

from which it follows that the waves of the first shape are displaced primarily along the free boundary, and the waves of the second shape along the boundary of separation.

3. By X and Y let us denote the projections of the vector of pressure forces applied to the contour of body C, and by M the moment of these pressure forces with respect to the origin of the coordinates. In order

to evaluate the mean values of X, Y and M during one period of oscillation we will use Kochin's formulas [1]

$$
\begin{aligned}
Y_{\text{cp}} + iX_{\text{cp}} &= \overset{\circ}{\rho}_2 S_{\text{cp}} - \frac{\overset{\circ}{\rho}_2}{4} \int\limits_{C_2} \left[\left(\frac{d\overset{\circ}{w}_{21}}{dz} \right)^2 + \left(\frac{d\overset{\circ}{w}_{22}}{dz} \right)^2 \right] dz \\
M_{\text{cp}} &= \overset{\circ}{\rho}_2 \, [x_c S]_{\text{cp}} - \frac{\overset{\circ}{\rho}_2}{4} \operatorname{Re} \left\{ \int\limits_{C_2} \left[\left(\frac{d\overset{\circ}{w}_{21}}{dz} \right)^2 + \left(\frac{d\overset{\circ}{w}_{22}}{dz} \right)^2 \right] z dz \right\}
\end{aligned} \tag{3.20}
$$

where S is the area bounded by the contour of body C, x_0 the abscissa of the center of gravity of this area, and C_2 any contour located in region $-1 > \operatorname{Im} z > -\infty$ which includes the contour C_1. The first terms on the right-hand side obviously depend upon the buoyant force. Since

$$
\frac{dw_{2m}^{\circ}}{dz} = \bar{v}_{2m}(z) = V_m(z) + U_m(z) \tag{3.21}
$$

and

$$
\int\limits_{C_2} V_m^2(z)\, dz = 0, \qquad \int\limits_{C_2} U_m^2(z)\, dz = 0
$$

then

$$
\int\limits_{C_2} \left(\frac{dw_{2m}^{\circ}}{dz} \right)^2 dz = 2 \int\limits_{C_2} \bar{v}_{2m}(z)\, U_m(z)\, dz
$$

From this, taking formulas (3.11) into account and changing the order of integration, we will obtain

$$
\begin{gathered}
\int\limits_{C_2} \left(\frac{d\overset{\circ}{w}_{21}}{dz} \right)^2 dz = \\
= 2 \int\limits_0^\infty H_1(\lambda)\, \overline{H}_1(\lambda) \left(-\frac{1}{2\pi} + G \right) d\lambda - 2E_0 H_1(\nu)\, \overline{H}_2(\nu) - 2G_0 H_1(\lambda_0)\, \overline{H}_2(\lambda_0) \\
\int\limits_{C_2} \left(\frac{d\overset{\circ}{w}_{22}}{dz} \right)^2 dz = \\
= 2 \int\limits_0^\infty H_2(\lambda)\, \overline{H}_2(\lambda) \left(-\frac{1}{2\pi} + G \right) d\lambda + 2E_0 H_2(\nu)\, \overline{H}_1(\nu) + 2G_0 H_2(\lambda_0)\, \overline{H}_1(\lambda_0)
\end{gathered} \tag{3.22}
$$

Introducing expressions (3.22) into the first formula (3.20), we find

$$
\begin{aligned}
Y_{\text{cp}} = \rho_2^{\circ} S_{\text{cp}} + \frac{\rho_2^{\circ}}{4\pi} \int\limits_0^\infty \left[|H_1(\lambda)|^2 + |H_2(\lambda)|^2 \right] d\lambda - \frac{\rho_2^{\circ}}{2} \int\limits_0^\infty \Big[|H_1(\lambda)|^2 + {} \\
+ |H_2(\lambda)|^2 \Big] G d\lambda
\end{aligned} \tag{3.23}
$$

$$
X_{\text{cp}} = \rho_2^{\circ} E_0 \operatorname{Im} [H_1(\nu)\, H_2(\nu)] + \rho_2^{\circ} G_0 \operatorname{Im} [H_1(\lambda_0)\, \overline{H}_2(\lambda_0)] \tag{3.24}
$$

We evaluate the moment M in the same manner. Near the point at infinity the functions $V_m(z)$ have the expansion:

$$V_m(z) = \frac{1}{2\pi i z}\int_{C_1} \bar{v}_{2m}(\zeta)\, d\zeta + \ldots = \frac{H_m(0)}{2\pi i z} + \ldots$$

consequently

$$\mathrm{Re}\Big[\int_{C_2} zV_m^2(z)\,dz\Big] = \mathrm{Re}\Big[\frac{H_m^2(0)}{2\pi i}\Big] = 0$$

Besides

$$\int_{C_1} zU_m^2(z)\,dz = 0$$

Therefore

$$M_{\mathrm{cp}} = \rho_2^\circ [x_c S]_{\mathrm{cp}} - \frac{\rho_2^\circ}{2}\mathrm{Re}\{\int_{C_1}[zV_1(z)U_1(z) + zV_2(z)U_2(z)]\,dz\} \tag{3.25}$$

or after evaluation

$$M_{\mathrm{cp}} = \rho_2^\circ\Big[x_c S\Big]_{\mathrm{cp}} + \frac{\rho_2^\circ}{2}\mathrm{Im}\Big\{\int_0^\infty\Big[\overline{H}_1(\lambda)\frac{dH_1}{d\lambda} + \overline{H}_2(\lambda)\frac{dH_2}{d\lambda}\Big]\Big(-\frac{1}{2\pi} + G\Big)d\lambda\Big\} +$$
$$+ \frac{\rho_2^\circ}{2}E_0\,\mathrm{Im}\Big[\overline{H}_1(\nu)\Big(\frac{dH_2}{d\lambda}\Big)_\nu - \overline{H}_2(\nu)\Big(\frac{dH_1}{d\lambda}\Big)_\nu\Big] + \frac{\rho_2^\circ}{2}G_0\,\mathrm{Im}\Big[\overline{H}_1(\lambda_0)\Big(\frac{dH_2}{d\lambda}\Big)_{\lambda_0} -$$
$$- \overline{H}_2(\lambda_0)\Big(\frac{dH_1}{d\lambda}\Big)_{\lambda_0}\Big] \tag{3.26}$$

In the absence of the upper layer of fluid it is possible to put $\kappa = 0$. In this case formulas (3.10), (3.16), (3.23), (3.24) and (3.26) will coincide with the corresponding Kochin [1] formulas.

For the evaluation of functions $H_m(\lambda)$ which express all the basic results of the problem, it is necessary to know the expressions of functions $V_{2m}(z)$ on the contour of body C. However, in case of comparatively large relative depth of immersion, it is possible to obtain fairly good approximation by introducing into formulas (3.8) for the functions $v_{2m}(z)$ their values $v_{2m\infty}(z)$, which correspond to the oscillation of the body in an unbounded liquid. It is possible to investigate a series of examples similar to those discussed by Kochin [1].

4. Solution of the problem with the aid of integral equations. 1. In accordance with the formulas derived in Section 3 it is possible to obtain exact expressions of values to be determined, if the solution of the problem given in Section 1 is known. Applying the boundary condition on the contour of the body C, this problem will be reduced to the solution of integral equations.

Assuming that the contour C is simple and with a continuous curvature, we will distribute along C pulsating sources with some density $[\gamma_1(\sigma)\cos t + \gamma_2(\sigma)\sin t]$, where σ is the length of the arc on the contour C, which corresponds to point $\zeta(\sigma)$ of this contour.

Then the complex potentials $w_{2m}^{\circ}(z)$ can be found in the form

$$w_{2m}^{\circ}(z) = \frac{1}{2\pi}\int\limits_C \gamma_m(\sigma)\ln[z-\zeta(\sigma)]\,d\sigma + w_{2m}^*(z) \tag{4.1}$$

where $w_{2m}^*(z)$ functions are analytical in the region E_2. Utilizing formulas (2.22), we obtain

$$w_{2m}^*(z) = \int\limits_C \Big\{\gamma_m(\sigma)\Big[\frac{1}{2\pi}\ln(z-\bar\zeta) + \int\limits_0^\infty Ge^{-i\lambda(z-\bar\zeta)}\frac{d\lambda}{\lambda}\Big] + E_m'(\nu)e^{-i\nu(z-\bar\zeta)} +$$
$$+ E_m'(\lambda_0)e^{-i\lambda_0(z-\bar\zeta)}\Big\}\,d\sigma \tag{4.2}$$

where

$$E_1'(\nu) = -\gamma_2(\sigma)\frac{E_0(\nu)}{\nu}, \qquad E_2'(\nu) = \gamma_1(\sigma)\frac{E_0(\nu)}{\nu}$$

$$E_1'(\lambda_0) = -\gamma_2(\sigma)\frac{E_0(\lambda_0)}{\lambda_0}, \qquad E_2'(\lambda_0) = \gamma_1(\sigma)\frac{E_0(\lambda_0)}{\lambda_0}$$

where we have denoted $E_0(\nu) = E_0$ and $E_0(\lambda_0) = G_0$.

It is obvious that conditions 1°-5° from Section 1 are thus satisfied.

Introducing expressions dw_{2m}°/dz from (4.1) into formulas (3.8), we obtain

$$H_m(\lambda) = i\int\limits_C \gamma_m(\sigma)e^{-i\lambda\zeta}d\sigma \tag{4.3}$$

Taking into account that

$$\int\limits_0^\infty Ge^{-i\lambda(z-\bar\zeta)}d\lambda = \int\limits_{L-} Ge^{-i\lambda(z-\bar\zeta)}d\lambda - iE_0(\nu)e^{-i\nu(z-\bar\zeta)} - iE_0(\lambda_0)e^{-i\lambda_0(z-\bar\zeta)}$$

(Fig. 2) and utilizing formula (4.3), after evaluation we obtain

$$\frac{dw_{2m}^{\circ}}{dz} = \int\limits_C \gamma_m(\sigma)\Big\{\frac{1}{2\pi}\Big[\frac{1}{z-\zeta}+\frac{1}{z-\bar\zeta}\Big] - i\int\limits_{L-} Ge^{-i\lambda(z-\bar\zeta)}d\lambda\Big\}\,d\sigma +$$
$$+ E_m(\nu)e^{-i\nu z} + E_m(\lambda_0)e^{-i\lambda_0 z} \tag{4.4}$$

where for values ν and λ_0, designated by Λ, we denote

$$E_1(\Lambda) = -iE_0(\Lambda)[\overline{H}_1(\Lambda) - i\overline{H}_2(\Lambda)],\ E_2(\Lambda), iE_1(\Lambda) \tag{4.5}$$

Functions $\gamma_m(\sigma)$ will be found from conditions 6^0 on the contour of body C, which can be written as follows

$$\mathrm{Re}\left[\frac{dw_{2m}{}^{\circ}}{dz} e^{i\theta}\right] = v_{nm}(s) \tag{4.6}$$

where θ is the angle between the exterior normal to the contour C and the x-axis.

Applying the formulas for the limiting value of the integral of Cauchy type on the contour C, we obtain from condition (4.6) an integral equation

$$\gamma_m(s) = -\int_C K(s, \sigma)\gamma_m(\sigma)\,d\sigma + f_m(s) \tag{4.7}$$

where

$$K(s, \sigma) = \mathrm{Re}\left\{\frac{1}{\pi}\left[\frac{e^{i\theta}}{z-\zeta} + \frac{e^{i\theta}}{z-\bar{\zeta}}\right] - 2ie^{i\theta}\int_{L-} Ge^{-i\lambda(z-\bar{\zeta})}\,d\lambda\right\} \tag{4.8}$$

$$f_m(s) = 2v_{nm}(s) - 2\,\mathrm{Re}\,\{E_m(\nu)e^{i(\theta-\nu z)} + E_m(\lambda_0)e^{i(\theta-\lambda_0 z)}\} \tag{4.9}$$

2. The problem has thus been reduced to the integral equation

$$\gamma(s) = \mu\int_C K(s, \sigma)\gamma(\sigma)\,d\sigma + f(s) \tag{4.10}$$

for $\mu = -1$. We will seek the solution for sufficiently small values of ν. Considering the limit at $\nu \to 0$, we obtain the equation

$$\gamma(s) = \mu\int_C K_0(s, \sigma)\gamma(\sigma)\,d\sigma + f_0(s) \tag{4.11}$$

where

$$K_0(s, \sigma) = \lim_{\nu\to 0} K(s, \sigma) = \frac{1}{\pi}\mathrm{Re}\left[\frac{e^{i\theta}}{z-\zeta} + \frac{e^{i\theta}}{z-(\bar{\zeta}-2i)}\right] = \frac{\cos(n, r)}{\pi r} + \frac{\cos(n, r')}{\pi r'}$$

$$f_0(s) = \lim_{\nu\to 0} f(s)$$

$$r = |z-\zeta|, \qquad r' = |z-(\bar{\zeta}-2i)|$$

By introducing contour C', which is symmetrical with the C relative to the line $y = -1$, we will rewrite equation (4.11) in the form

$$\gamma(s) = \mu\int_{C+C'} \frac{\cos(n, r)}{\pi r}\gamma(\sigma)\,d\sigma + f_0(s) \tag{4.12}$$

considering functions $\gamma(s)$ and $f_0(s)$ to have equal values at points symmetrical relative to line $y = -1$.

But it is known that equation (4.12) has a simple characteristic number $\mu = 1$, and that all other characteristic numbers μ_k are such that $|\mu_k| > 1$. Because the homogeneous equation conjugate to (4.12) has a unique independent solution $\gamma(\sigma) \equiv 1$ at $\mu = 1$, the condition for a solution of equation (4.12) to exist at $\mu = 1$ will be that the equation

$$\int_C f_0(s)\,ds = 0$$

be satisfied; this is achieved if the contour C is assumed not to deform during oscillations. Therefore, the solution of equation (4.12) will be a meromorphic function of μ, where the poles of this function are located outside of a circle $|\mu| \leqslant 1$. But then, for sufficiently small values of ν the solution of equation (4.10) will have only one characteristic number within the circle $|\mu| \leqslant R$, where $R > 1$. We will show that it is equal to unity. In fact, we have

$$\int_C K(s, \sigma)\,ds = \int_C \frac{\cos(n, r)}{\pi r}\,ds + \mathrm{Re}\Big[\int_C g(z)\,dz\Big] = \int_C \frac{\cos(n, r)}{\pi r}\,ds = 1 \qquad (4.13)$$

where $g(z)$ is an analytical function in the region E_2. Therefore, the homogeneous equation conjugate to (4.10) has the solution $\gamma(\sigma) \equiv 1$, at $\mu = 1$, i.e. the number $\mu = 1$ is a characteristic number. The condition of solution of equation (4.10) at $\mu = 1$, having the form

$$\int_C f(s)\,ds = 0 \qquad (4.14)$$

is achieved, provided that

$$\int_C v_{nm}(s)\,ds = 0, \qquad \mathrm{Re}\,[\int_C e^{i(\theta - \lambda z)}\,ds] = \mathrm{Re}\,[\int_C e^{-i\lambda z}\,dz] = 0$$

Consequently, for a sufficiently small ν, the solution of equation (4.10) exists and is a meromorphic function of μ, and the characteristic number $\mu = 1$ is not the pole of this function. Since for sufficiently small values of ν all poles of the function $\gamma(s)$ are situated outside circle $|\mu| \leqslant 1$, the solution of equation (4.10) can be taken in the form of series with exponent μ, which converges in a circle $|\mu| \leqslant 1$.

Hence,

$$\gamma(s) = \sum_{l=0}^{\infty} \mu^l q_l(s) \qquad (4.15)$$

For $\mu = -1$ we obtain the solutions of our equations (4.7):

$$\gamma_m(s) = \sum_{l=0}^{\infty} (-1)^l q_{ml}(s) \qquad (4.16)$$

where

$$q_{m0}(s) = f_m(s), \qquad q_{ml}(s) = \int_C K(s, \sigma)\, q_{m,\, l-1}(\sigma)\, d\sigma \tag{4.17}$$

3. Let us find the constants $E_m(\nu)$ and $E_m(\lambda_0)$ through which functions $f_m(s)$ are expressed, and consequently also the solution (4.16). The last can be shown in the form

$$\gamma_m(s) = \gamma_m{}^\circ(s) + \mathrm{Re}\,[E_m(\nu)\,\gamma(s, \nu)] + \mathrm{Re}\,[E_m(\lambda_0)\,\gamma(s, \lambda_0)] \tag{4.18}$$

where $\gamma_m{}^\circ(s)$ and $\gamma(s, \lambda)$ are the solutions of equations

$$\gamma_m{}^\circ(s) = -\int_C K(s, \sigma)\,\gamma_m{}^\circ(\sigma)\, d\sigma + 2v_{nm}(s) \tag{4.19}$$

$$\gamma(s, \lambda) = -\int_C K(s, \sigma)\,\gamma(\sigma, \lambda)\, d\sigma - 2e^{i\vartheta - i\lambda z} \tag{4.20}$$

where λ assumes values of ν and λ_0.

Introducing expressions (4.18) into formulas (4.3) and using the obvious equality $\mathrm{Re}\,[a \cdot b] + i\,\mathrm{Re}\,[ia \cdot b] = a \cdot b$, we find that

$$H_1(\Lambda) + iH_2(\Lambda) = H_1{}^\circ(\Lambda) + iH_2{}^\circ(\Lambda) + \overline{E_1(\nu)}\, H(\nu, \Lambda) + \overline{E_1(\lambda_0)}\, H(\lambda_0, \Lambda) \tag{4.21}$$

where for values λ and Λ, becoming ν and λ_0, we denote

$$H_m{}^\circ(\Lambda) = i\int_C \gamma_m{}^\circ(\sigma)\, e^{-\Lambda\zeta}\, d\sigma, \qquad H(\lambda, \Lambda) = i\int_C \overline{\gamma(\sigma, \lambda)}\, e^{-i\Lambda\zeta}\, d\sigma \tag{4.22}$$

Introducing in formulas (4.5) expressions (4.21) at $\Lambda = \nu$ and $\Lambda = \lambda_0$, we will have to find two equations for $E(\nu)$ and $E_1(\lambda_0)$, from which we will find

$$E_1(\nu) = -\frac{\alpha_1}{\beta}, \qquad E_1(\lambda_0) = -\frac{\alpha_2}{\beta}, \qquad E_2(\nu) = iE_1(\nu), \qquad E_2(\lambda_0) = iE_1(\lambda_0) \tag{4.23}$$

where

$$\alpha_1 = E_0(\nu)\, E_0(\lambda_0)\, \overline{H(\lambda_0, \nu)}\, [\overline{H_1{}^\circ(\lambda_0) - iH_2{}^\circ(\lambda_0)}] + $$
$$+ iE_0(\nu)\,[1 + iE_0(\lambda_0)\, \overline{H(\lambda_0, \lambda_0)}]\,[\overline{H_1{}^\circ(\nu) - iH_2{}^\circ(\nu)}]$$

$$\alpha_2 = E_0(\nu)\, E_0(\lambda_0)\, \overline{H(\nu, \lambda_0)}\, [\overline{H_1{}^\circ(\nu) - iH_2{}^\circ(\nu)}] + $$
$$+ iE_0(\lambda_0)\,[1 + iE_0(\nu)\, \overline{H(\nu, \nu)}]\,[\overline{H_1{}^\circ(\lambda_0) - iH_2{}^\circ(\lambda_0)}]$$

$$\beta = E_0(\nu)\, E_0(\lambda_0)\, \overline{H(\nu, \lambda_0)}\, \overline{H(\lambda_0, \nu)} + [1 + iE_0(\nu)\, \overline{H(\nu, \nu)}]\,[1 + iE_0(\lambda_0)\, \overline{H(\lambda_0, \lambda_0)}] \tag{4.24}$$

Using equations (4.7) and equalities (4.13) and (4.14) it is not difficult to prove equalities

$$\int_C \gamma_m(\sigma)\, d\sigma = 0 \tag{4.25}$$

Using (4.25) we obtain from (4.22) at $\nu = 0$

$$[H_m^\circ(\Lambda)]_{\Lambda=0} = 0, \qquad [H(\lambda, \Lambda)]_{\lambda=\Lambda=0} = 0$$

from which it follows that for sufficiently small values of ν the denominator β in formulas (4.23) is different from zero.

Determining the values of $E_m(\nu)$ and $E_m(\lambda_0)$ from formulas (4.23) we can then find functions $\gamma_m(s)$ from (4.18). Introducing the latter into formulas (4.3) we find functions $H_m(\lambda)$, which permit the determination of the principal values of the problem according to the formulas in Section 3. Considering the case of a sufficiently small length (compared to unity) of contour C, it is possible to prove the convergence of the process of successive approximations at any value of parameter ν by application of the principle of transformations of decreasing scale.

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